# Instability of a two-dimensional extremal black hole

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## Abstract

We consider the perturbation of tachyon about the extremal ground state of a two-dimensional (2D) electrically charged black hole. It is found that the presenting potential to on-coming tachyonic wave takes a double-humped barrier well. This allows an exponentially growing mode with respect to time. This extremal ground state is classically unstable. We conclude that the 2D extremal electrically charged black hole cannot be a candidate for the stable endpoint of the Hawking evaporation.

Recently the extremal black holes have received much attention. Extremal black hole provide a simple laboratory in which to investigate the quantum aspects of black hole [1]. One of the crucial features is that the Hawking temperature vanishes. The extremal magnetically charged black holes are shown to be classically as well as quantum mechanically stable [2]. The black hole with $M > Q$ will tend to Hawking radiate down to its extremal $M = Q$ state. Thus the extremal black hole may play a role of the stable endpoint for the Hawking evaporation. It has been more recently proposed that although the extremal black hole has nonzero area, it has zero entropy [3]. This is because the extremal case is distinct topologically from the nonextremal one. For example, the extremal black hole has the infinite throat. The origin is effectively removed from the manifold. The topology is no longer a disk but rather, an annulus whose inner boundary is at infinite distance.

In this letter, we will address the stability aspects of the 2D extremal electrically charged black hole. It is the fundamental test for the 2D extremal black hole to exist. We always visualize the black hole as presenting an effective potential barrier (or well) to the on-coming waves [4]. One easy way of understanding the attributes of an extremal black hole is to find out how it reacts to external perturbations. In deciding whether or not the the extremal black hole is stable, one starts with a physical perturbation which is regular everywhere in space at the initial time $t = 0$ [5]. And then see whether such a perturbation will grow with time. If there exists an exponentially growing mode, the extremal black hole is unstable.

We will work with part of the low energy action to heterotic string theory [6]

$$S_{l-e} = \int d^2x \sqrt{-G} e^{-2\Phi} \{ R + 4(\nabla\Phi)^2 + \alpha^2 - \frac{1}{2}F^2 - \frac{1}{2}(\nabla T)^2 + T^2 \}. \tag{1}$$

Here are all string fields ( metric $G_{\mu\nu}$, dilaton $\Phi$, Maxwell field $F_{\mu\nu}$, and tachyon $T$). Setting $\alpha^2 = 8$ and after deriving equations, we take the transformation

$$-2\Phi \rightarrow \Phi, \quad T \rightarrow \sqrt{2}T, \quad -R \rightarrow R. \tag{2}$$

Then the equations of motion become

$$R_{\mu\nu} + \nabla_\mu\nabla_\nu\Phi + F_{\mu\rho}F_\nu{}^\rho + \nabla_\mu T\nabla_\nu T = 0, \tag{3}$$

$$(\nabla\Phi)^2 + \nabla^2\Phi - \frac{1}{2}F^2 - 2T^2 - 8 = 0, \tag{4}$$

$$\nabla_\mu F^{\mu\nu} + (\nabla_\mu\Phi)F^{\mu\nu} = 0, \tag{5}$$

$$\nabla^2 T + \nabla\Phi\nabla T + 2T = 0. \tag{6}$$

An electrically charged black hole solution to the above equations is given by

$$\bar{\Phi} = 2\sqrt{2}r, \quad \bar{F}_{tr} = Qe^{-2\sqrt{2}r}, \quad \bar{T} = 0, \quad \bar{G}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix}, \tag{7}$$

with

$$f = 1 - \frac{2M}{2\sqrt{2}}e^{-2\sqrt{2}r} + \frac{Q^2}{8}e^{-4\sqrt{2}r}, \tag{8}$$

where $M$ and $Q$ are the mass and charge of the black hole, respectively. For convenience, we take $M = \sqrt{2}$. For $0 < |Q| < M$, the double horizons $(r_\pm)$ are given by

$$r_\pm = \frac{1}{2\sqrt{2}}\log\left[\frac{1 \pm \sqrt{1 - \frac{Q^2}{2}}}{2}\right], \tag{9}$$

where $r_+(r_-)$ correspond to the event (Cauchy) horizons. This charged black hole may provide an ideal setting for studying the late stages of Hawking evaporation. For $Q = M$, two horizons coincide :$r_+ = r_- \equiv r_o$. We are here interested in this extremal limit.

To study the propagation of string fields, we introduce small perturbation fields around the background solution as [7]

$$F_{tr} = \bar{F}_{tr} + \mathcal{F}_{tr} = \bar{F}_{tr}[1 - \frac{\mathcal{F}(r,t)}{Q}], \tag{10}$$

$$\Phi = \bar{\Phi} + \phi(r,t), \tag{11}$$

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + h_{\mu\nu} = \bar{G}_{\mu\nu}[1 - h(r,t)], \tag{12}$$

$$T = \bar{T} + \tilde{t} \equiv \exp(-\frac{\bar{\Phi}}{2})[0 + t(r,t)]. \tag{13}$$

One has to linearize (3)-(6) in order to obtain the equations governing the perturbations. It is important to check whether the graviton $(h)$, dilaton $(\phi)$, Maxwell mode $(\mathcal{F})$ and tachyon $(t)$ are physically propagating modes in the 2D charged black hole background. We review

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the conventional counting of degrees of freedom. The number of degrees of freedom for the gravitational field $(h_{\mu\nu})$ in $D$-dimensions is $(1/2)D(D-3)$. For a Schwarzschild black hole, we obtain two degrees of freedom. These correspond to the Regge-Wheeler mode for odd-parity perturbation and Zerilli mode for even-parity perturbation [4]. We have $-1$ for $D = 2$. This means that in two dimensions the contribution of the graviton is equal and opposite to that of a spinless particle (dilaton). The graviton-dilaton modes $(h + \phi, h - \phi)$ are gauge degrees of freedom and thus turn out to be nonpropagating modes[6]. In addition, the Maxwell field has $D-2$ physical degrees of freedom. The Maxwell field has no physical degrees of freedom for $D = 2$. Actually from (5) it turns out to be a redundant one $(\mathcal{F} = -Q(h + \phi))$. Since these all are nonpropagating modes, it is not necessary to linearize (3)-(4). The stability should be based on the physical degrees of freedom. The tachyon is a physically propagating mode. Its linearized equation is

$$f^2 t'' + f f' t' - [\sqrt{2} f f' - 2f(1 - f)]t - \partial_t^2 t = 0, \tag{14}$$

where the prime ($\prime$) denotes the derivative with respect to $r$. To study the stability, the above equation should be transformed into the one-dimensional Schrödinger equation. Introducing the coordinate transformation

$$r \to r^* \equiv g(r),$$

(14) can be rewritten as

$$f^2 g'^2 \frac{\partial^2}{\partial r^{*2}} t + f\{f g'' + f' g'\} \frac{\partial}{\partial r^*} t - [\sqrt{2} f f' - 2f(1 - f)]t - \frac{\partial^2}{\partial t^2} t = 0. \tag{15}$$

Requiring that the coefficient of the linear derivative vanish, one finds the relation

$$g' = \frac{1}{f}. \tag{16}$$

Assuming $t(r^*, t) \sim \tilde{t}(r^*)e^{i\omega t}$, one can cast (15) into one-dimensional Schrödinger equation

$$\{\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\}\tilde{t} = 0, \tag{17}$$

where the effective potential $V(r)$ for the extremal case $(Q = M = \sqrt{2})$ is given by

$$V(r) = f(\sqrt{2}f' - 2(1-f)) = 2e^{-2\sqrt{2}r}(1 - \frac{1}{2}e^{-2\sqrt{2}r})^2(1 - \frac{3}{4}e^{-2\sqrt{2}r}). \tag{18}$$

The event horizon is located at $r_o = -0.245$. As is shown in Fig.1, we have a double-humped barrier well outside the black hole. According to the analysis of potentials, a double-humped potential appears when the nonextremal black hole (a simple barrier) approaches the extremal one. Now let us translate the potential $V(r)$ into $V(r^*)$. With $f = (1 - \frac{1}{2}\exp(-2\sqrt{2}r))^2$, one obtain the explicit form of $r^*$

$$r^* = r - \frac{1}{2\sqrt{2}(1 - \frac{1}{2}e^{-2\sqrt{2}r})} + \frac{1}{2\sqrt{2}}\log|1 - \frac{1}{2}e^{-2\sqrt{2}r}|. \tag{19}$$

Since both the forms of $V(r)$ and $r^*$ are very complicated, we are far from obtaining the exact form of $V(r^*)$. Instead we can find an approximate form. From (19), in the asymptotically flat region one finds that $r^* \simeq r$. (18) takes the asymptotic form

$$V_{r* \to \infty} \simeq 2\exp(-2\sqrt{2}r^*). \tag{20}$$

On the other hand, near the horizon $(r = r_o)$ one has

$$r^* \simeq -\frac{1}{2\sqrt{2}(1 - e^{-2\sqrt{2}r})}. \tag{21}$$

Approaching the horizon $(r \to r_o, r^* \to -\infty)$, the potential takes the form

$$V_{r* \to -\infty} = -\frac{1}{4r^{*2}}. \tag{22}$$

Using (20) and (22) one can construct the approximate form $V_{app}(r^*)$ (Fig. 2). This is also a double-humped barrier well which is localized at the origin of $r^*$. Our stability analysis is based on the equation

$$\{\frac{d^2}{dr^{*2}} + \omega^2 - V_{app}(r^*)\}\tilde{t} = 0. \tag{23}$$

As we have probably guessed, two kinds of solutions to the Schrödinger equation correspond to the bound and scattering states. In our case $V_{app}(r^*)$ admits two solutions depending on the signs of the energy.

(i) For $E > 0(\omega = \text{real})$, the asymptotic solution for $\tilde{t}$ is given by

$$\tilde{t}_\infty = \exp(i\omega r^*) + R\exp(-i\omega r^*) \qquad (r^* \to \infty), \tag{24}$$

$$\tilde{t}_{EH} = T\exp(i\omega r^*) \qquad (r^* \to -\infty), \tag{25}$$

where $R$ and $T$ are the scattering amplitudes of two waves which are reflected and trans-mitted by the potential $V_{app}(r^*)$, when a wave of unit amplitude with the frequency $\omega$ is incident on the black hole from infinity.

(ii) For $E < 0(\omega = -i\alpha, \alpha$ is positive and real), we have the bound state. Eq. (23) and possible asymptotic solutions are given by

$$\frac{d^2}{dr^{*2}}\tilde{t} = (\alpha^2 + V_{app}(r^*))\tilde{t}, \tag{26}$$

$$\tilde{t}_\infty \sim \exp(\pm\alpha r^*), \qquad (r^* \to \infty) \tag{27}$$

$$\tilde{t}_{EH} \sim \exp(\pm\alpha r^*) \qquad (r^* \to -\infty). \tag{28}$$

To ensure that the perturbation falls off to zero for large $r^*$, we choose $\tilde{t}_\infty \sim \exp(-\alpha r^*)$. In the case of $\tilde{t}_{EH}$, the solution $\exp(\alpha r^*)$ goes to zero as $r^* \to -\infty$. Now let us observe whether or not $\tilde{t}_{EH} \sim \exp(\alpha r^*)$ can be matched to $\tilde{t}_\infty \sim \exp(-\alpha r^*)$. Assuming $\tilde{t}$ to be positive, the sign of $d^2\tilde{t}/dr^{*2}$ can be changed from $+$ to $-$ as $r^*$ goes from $\infty$ to $-\infty$. If we are to connect $\tilde{t}_{EH}$ at one end to a decreasing solution $\tilde{t}_\infty$ at the other, there must be a point $(d^2\tilde{t}/dr^{*2} < 0, d\tilde{t}/dr^* = 0)$ at which the signs of $\tilde{t}$ and $d^2\tilde{t}/dr^{*2}$ are opposite : this is compatible with the shape of $V_{app}(r^*)$ in Fig.2. It thus is possible for $\tilde{t}_{EH}$ to be connected to $\tilde{t}_\infty$ smoothly. Therefore a bound state solution is given by

$$\tilde{t}_\infty \sim \exp(-\alpha r^*), \qquad (r^* \to \infty) \tag{29}$$

$$\tilde{t}_{EH} \sim \exp(\alpha r^*) \qquad (r^* \to -\infty). \tag{30}$$

This is a regular solution everywhere in space at the initial time $t = 0$. However, $\omega = -i\alpha$ implies $t_\infty(r^*, t) = \tilde{t}_\infty(r^*)\exp(-i\omega t) \sim \exp(-\alpha r^*)\exp(\alpha t)$ and $t_{EH}(r^*, t) = \tilde{t}_{EH}(r^*)\exp(-i\omega t) \sim \exp(\alpha r^*)\exp(\alpha t)$. This means that there exists an exponentially grow-ing mode with time. Therefore, the 2D extremal ground state is classically unstable. The

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origin of this instability comes from a double-humped barrier well. This potential appears when the nonextremal black hole approaches the extremal limit. As is discussed in Ref. [8], the quantum stress tensor of a scalar field (instead of the tachyon) in the extremal black hole diverges at the horizon. This means that the extremal black hole is quantum-mechanically unstable. This divergence can be better understood by the regarding an extremal black hole as the limit of a nonextremal one. A nonextremal black hole has an outer (event) and an inner (Cauchy) horizon, and these come together in the extremal limit. In this case, we find that if we adjust the quantum state of the scalar field so that the stress tensor is finite at the outer horizon, it always diverges at the inner horizon. Thus it is not so surprising that in the extremal limit ( when the two horizons come together) the divergence persists, although it has a softened form. By the similar way, it conjectures that the classical instability originates from the instability (blueshift) of the inner horizon [7].

In conclusion, the 2D extremal electrically charged black hole cannot be a candidate for the stable endpoint of the Hawking evaporation.

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Fig.1 : The $M = Q = \sqrt{2}$ graph of the effective potential of tachyon $(V(r))$. The event horizon is at $r_o = -0.245$. This takes a double-humped barrier well outside the black hole.

Fig.2 :The approximate form of potential$(V_{app}(r^*))$ outside black hole. This also takes a double-humped barrier well. This is localized at $r^* = 0$, falls to zero exponentially as $r^* \to \infty$ and inverse-squarely as $r^* \to -\infty$ (solid lines). The dotted line is used to connect two boundaries.